TILTING BUNDLES ON SOME RATIONAL SURFACES

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1. Introduction

Let X be a smooth projective variety defined over an algebraically closed field k, and let $\mathrm{D}^{\mathrm{b}}(X) = \mathrm{D}^{\mathrm{b}}(\mathcal{O}_X\text{-mod})$ be the derived category of bounded complexes of coherent sheaves of $\mathcal{O}_X$-modules. A natural question is: when is $\mathrm{D}^{\mathrm{b}}(X)$ freely and finitely generated? This was shown to be the case when X is a projective space, by Beilinson [Be1], and when X is a quadric, Grassmannian or flag variety, by Kapranov [Ka]. In this paper, we describe a method for attacking this problem and illustrate it with the examples of some rational surfaces. In fact, it is now known from general methods of Orlov [Or] that $\mathrm{D}^{\mathrm{b}}(X)$ freely and finitely generated for all rational surfaces X.

This paper elaborates the view-point of Bondal [Bo], who observed that showing that $\mathrm{D}^{\mathrm{b}}(X)$ is free and finitely generated by a sheaf $T \in \mathcal{O}_X$-mod amounts to showing that $\mathrm{D}^{\mathrm{b}}(X)$ is equivalent as a triangulated category to the derived category $\mathrm{D}^{\mathrm{b}}(A) = \mathrm{D}^{\mathrm{b}}(\text{mod-}A)$ of finite dimensional right modules over the finite dimensional algebra $A = \mathrm{Hom}_X(T,T)$. The equivalence is provided explicitly by the pair of adjoint functors

$$\begin{aligned} -\overset{\mathbf{L}}{\otimes}_A T &\ :\ \mathrm{D}^{\mathrm{b}}(A) \longrightarrow \mathrm{D}^{\mathrm{b}}(X) \\ \mathbf{R}\mathrm{Hom}_X(T,-) &\ :\ \mathrm{D}^{\mathrm{b}}(X) \longrightarrow \mathrm{D}^{\mathrm{b}}(A) \end{aligned}$$

Following the terminology of representation theory (cf. [Ba]), the sheaf T is called a tilting sheaf or, when it is locally free, a tilting bundle. The precise definition is as follows.

Definition 1.1. A *tilting sheaf* is a sheaf $T \in \mathcal{O}_X$-mod for which
(i) $\mathrm{Ext}^i_X(T,T) = 0$ for $i \geq 1$,
(ii) the algebra $A = \mathrm{Hom}_X(T,T)$ has finite global dimension,
(iii) T generates the derived category $\mathrm{D}^{\mathrm{b}}(\mathcal{O}_X\text{-mod})$.
If T satisfies just the first two conditions then it is called a *partial tilting sheaf.*

The search for a tilting sheaf is naturally divided into two steps. First find a partial tilting sheaf T with the correct number of summands; this number being the rank of $K_0(X)$. Second, show that T generates the derived category. The first step appears more 'mechanical' than the

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second, since it mainly involves calculating cohomology. However, the principal observation of this paper is that, at least when T is bundle, the second step can be made similarly mechanical.

Theorem 1.2. *Let X be a smooth projective variety and T be a partial tilting bundle with* $\mathrm{Hom}_X(T,T) = A$. *Then T is a tilting bundle if and only if the natural map $T^\vee \overset{\mathbf{L}}{\boxtimes}_A T \to \mathcal{O}_\Delta$ is an isomorphism in* $\mathrm{D^b}(\mathcal{O}_{X\times X}\text{-mod})$. *Furthermore, this map is an isomorphism if the fibres T_x for $x \in X$, regarded as left A-modules, satisfy the following conditions*
i) for all x, $\mathrm{Hom}_A(T_x,T_x) = k$ *and* $\mathrm{Ext}^i_A(T_x,T_x) = 0$ *for* $i > \dim X$,
ii) for $x \neq y$, $\mathrm{Hom}_A(T_x,T_y) = 0$ *and* $\mathrm{Ext}^i_A(T_x,T_y) = 0$ *for* $i \geq 1$.

The paper is laid out as follows. In §2 we recall the standard definitions and theorems for tilting sheaves. In §3–§5 we describe the techniques we use in the paper to identify tilting bundles. In §6–§8 we find tilting bundles on the rational surfaces mentioned above. We conclude in §9 with some remarks and conjectures about possible further developments.

Notation and Conventions. All varieties and algebras are defined over a fixed algebraically closed field k. In all categories morphisms act on the left. A 'sheaf' on X is a coherent sheaf of $\mathcal{O}_X$-modules. We shall not distinguish between a bundle and its locally-free sheaf of sections. The dual of a bundle T is denoted $T^\vee$. When X is smooth, we write $\mathrm{D^b}(X)$ for $\mathrm{D^b}(\mathcal{O}_X\text{-mod})$, and when A has finite global dimension, we write $\mathrm{D^b}(A)$ for $\mathrm{D^b}(\text{mod-}A)$.

2. Tilting sheaves

In this section, we recall the definitions and basic theorems (with sketched proofs) for tilting sheaves, based on similar definitions and results in the theory of tilting between finite dimensional algebras. See [Ba] for more details.

Recall that an algebra A has finite global dimension, equal to d, if and only if $\mathrm{Ext}^i(M,N) = 0$, for all $i > d$ and for all $M, N \in \text{mod-}A$. Furthermore, T generates $\mathrm{D^b}(\mathcal{O}_X\text{-mod})$ if and only if the latter is equivalent to its smallest triangulated subcategory which contains all the summands of T. Since there is no loss of generality in assuming that the indecomposable summands of T are pairwise non-isomorphic, we will make this assumption in future.

Theorem 2.1. *Let T be a partial tilting sheaf and* $A = \mathrm{End}(T)$. *Then the derived functor*

$$\mathbf{R}\mathrm{Hom}(T,-) : \mathrm{D^b}(\mathcal{O}_X\text{-mod}) \longrightarrow \mathrm{D^b}(\text{mod-}A).$$

is a left inverse of the derived functor

$$- \overset{\mathbf{L}}{\otimes}_A T : \mathrm{D^b}(\text{mod-}A) \longrightarrow \mathrm{D^b}(\mathcal{O}_X\text{-mod})$$

Hence, these functors define an equivalence between $\mathrm{D^b}(\text{mod-}A)$ *and the triangulated subcategory of* $\mathrm{D^b}(\mathcal{O}_X\text{-mod})$ *generated by* T*.*

Proof. Observe that $A \overset{\mathbf{L}}{\otimes}_A T = T$ and $\mathbf{R}\mathrm{Hom}(T,T) = A$. Hence, the composite functor $\mathbf{R}\mathrm{Hom}(T, - \overset{\mathbf{L}}{\otimes}_A T)$ is the identity on A. But, since A has finite global dimension, it generates $\mathrm{D^b}(\text{mod-}A)$ and so the composite is the identity on $\mathrm{D^b}(\text{mod-}A)$. □

From this there immediately follows

Theorem 2.2. *If* T *is a tilting sheaf, then the functors* $\mathbf{R}\mathrm{Hom}(T,-)$ *and* $- \overset{\mathbf{L}}{\otimes}_A T$ *are mutually inverse equivalences between* $\mathrm{D^b}(\mathcal{O}_X\text{-mod})$ *and* $\mathrm{D^b}(\text{mod-}A)$*.*

The existence of a tilting sheaf puts rather a strong restriction on X, namely that its Grothendieck group $K_0(X) = K_0(\mathcal{O}_X\text{-mod})$ is isomorphic to $\mathbb{Z}^n$.

Corollary 2.3. *Suppose* X *has a tilting sheaf* T *with non-isomorphic indecomposable summands* $T_1, \cdots, T_n$*. Then* $K_0(\mathcal{O}_X\text{-mod})$ *is freely generated by the classes* $[T_1], \ldots, [T_n]$*.*

Proof. The derived equivalence induces an isomorphism between $K_0(\mathcal{O}_X\text{-mod})$ and $K_0(\text{mod-}A)$ under which $[T_1], \ldots, [T_n]$ correspond to the classes of the indecomposable projective A-modules, which form a basis for $K_0(\text{mod-}A)$. □

3. Tilting Bundles and Resolutions of the Diagonal

In general, it is not so clear how to check whether a sheaf T generates the derived category $\mathrm{D^b}(\mathcal{O}_X\text{-mod})$, i.e. Condition (iii) in Definition 1.1. In this section, we give an equivalent condition, which can be verified more easily, at least when T is a vector bundle, provided one can calculate derived tensor products. We give a method for doing this in Section 5.

First observe that, for any $T \in \mathcal{O}_X\text{-mod}$, there is a natural map

$$\mathrm{Hom}(T,E) \otimes_A T \to E,$$

for all $E \in \mathcal{O}_X\text{-mod}$, and hence, taking derived functors, there is also a natural map

$$\eta_E : \mathbf{R}\mathrm{Hom}(T,E) \overset{\mathbf{L}}{\otimes}_A T \longrightarrow E,$$

for all $E \in \mathrm{D^b}(\mathcal{O}_X\text{-mod})$. From Theorem 2.1, we see that a partial tilting sheaf T is actually a tilting sheaf if and only if η_E is always an isomorphism. The next result shows that it is essentially sufficient to check that this map is an isomorphism when $E = \mathcal{O}_x$, the structure sheaf of a point x, for all $x \in X$.

Notation: $\mathcal{O}_\Delta$ is the structure sheaf of the diagonal $\Delta \subseteq X \times X$, to be thought of as the family $\{\mathcal{O}_x\}_{x\in X}$, via the first projection π_1 :

$X \times X \to X$; $A \boxtimes B$ is the exterior tensor product of A and B in $\mathcal{O}_X$-mod, i.e. $A \boxtimes B = \pi_1^* A \otimes \pi_2^* B$ in $\mathcal{O}_{X\times X}$-mod.

Proposition 3.1. *Let T be a partial tilting bundle with* $\mathrm{End}(T) = A$. *Then T is a tilting bundle if and only if the natural map $T^\vee \overset{\mathbf{L}}{\boxtimes}_A T \to \mathcal{O}_\Delta$ is an isomorphism in* $\mathrm{D^b}(\mathcal{O}_{X\times X}\text{-mod})$.

Proof. Suppose T is a vector bundle. Then $\mathbf{R}\mathrm{Hom}(T, \mathcal{O}_x) = T_x^\vee$ and so there are natural maps $\eta_x : T_x^\vee \overset{\mathbf{L}}{\otimes}_A T \to \mathcal{O}_x$, for each $x \in X$. These fit together to give the natural map

$$\eta_\Delta : T^\vee \overset{\mathbf{L}}{\boxtimes}_A T \longrightarrow \mathcal{O}_\Delta,$$

which can be represented by a complex on $X \times X$ whose last term is $\mathcal{O}_\Delta$. If T is a tilting bundle, then each η_x is an isomorphism, i.e. the complex is exact on each fibre of π_1, hence it is exact, i.e. η_Δ is an isomorphism.

On the other hand, observe that

$$\mathbf{R}\mathrm{Hom}(T, E) \overset{\mathbf{L}}{\otimes}_A T = \mathbf{R}\pi_{2*}\left(\pi_1^* E \overset{\mathbf{L}}{\otimes} \pi_1^* T^\vee \overset{\mathbf{L}}{\otimes}_A \pi_2^* T\right)$$

while taking derived functors of the equation $E = \pi_{2*}(\pi_1^* E \otimes \mathcal{O}_\Delta)$ yields

$$E = \mathbf{R}\pi_{2*}\left(\pi_1^* E \overset{\mathbf{L}}{\otimes} \mathcal{O}_\Delta\right).$$

Furthermore, the natural map η_E between the left-hand-sides of the two equations above, is induced by η_Δ acting on the right-hand-sides. Hence, if η_Δ is an isomorphism in $\mathrm{D^b}(\mathcal{O}_{X\times X}\text{-mod})$, then η_E is an isomorphism in $\mathrm{D^b}(\mathcal{O}_X\text{-mod})$ for all E, and hence T is a tilting bundle. ☐

4. Exceptional Sheaves and Collections

It can also be a little difficult, in general, to check whether a finite dimensional algebra A has finite global dimension, i.e. Condition (ii) of Definition 1.1. However, there is a simple criterion which is sufficient for the cases which we shall encounter in this paper. This criterion states that any 'triangular' algebra has finite global dimension [BB]. An algebra is triangular if its indecomposable projective modules $P_1, \dots, P_n$ all satisfy $\mathrm{Hom}(P_i, P_i) = k$ and can be ordered in such a way that $\mathrm{Hom}(P_j, P_i) = 0$, if $i < j$. If $A = \mathrm{End}(T)$, then these conditions are equivalent to those obtained by replacing $P_1, \dots, P_n$ by the indecomposable summands of T. Combining these conditions with Condition (i) of Definition 1.1, we recover the notion of a 'strongly exceptional collection' of sheaves. ([DL],[GR], [Bo]). Recall

Definition 4.1.
i) A sheaf E is *exceptional* if $\mathrm{Hom}(E, E) = k$ and $\mathrm{Ext}^i(E, E) = 0$, for $i \geq 1$,

ii) An ordered collection $E_1, \dots, E_n$ of sheaves is *strongly exceptional* if each E_i is exceptional, $\operatorname{Hom}(E_k, E_j) = 0$ for $j < k$, and $\operatorname{Ext}^i(E_j, E_k) = 0$ for $i \geq 1$ and all j, k.

From the remarks above we immediately obtain

Lemma 4.2. *If $E_1, \dots, E_n$ is a strongly exceptional collection, then $E_1 \oplus \cdots \oplus E_n$ is a partial tilting sheaf.*

In this paper, we shall make do with strongly exceptional collections of line bundles, for which Definition 4.1 immediately gives

Lemma 4.3. *A collection of line bundles $L_1, \dots, L_n$ on a variety X is strongly exceptional if*
i) $H^i(\mathcal{O}_X) = 0$, for all $i > 0$
and each difference (in $\operatorname{Pic} X$*) $D = L_j^\vee \otimes L_k$, for $j < k$, satisfies*
ii) $H^i(D) = 0$, for $i > 0$, and $H^i(D^\vee) = 0$, for all i.

Note that Condition (i) is birationally invariant and, in particular, is therefore satisfied by any rational surface, because it is satisfied by $\mathbb{P}^2$. Condition (ii) emphasises the fact that it is the differences that make a collection exceptional.

Remark 4.4. A strongly exceptional collection which generates the derived category, i.e. which makes up a tilting sheaf, is the essential ingredient, i.e. the 'foundation' (or sometimes 'thread'), of a helix (see [Bo] Theorem 4.1). This notion was introduced by Gorodentsev & Rudakov [GR] to study vector bundles on $\mathbb{P}^n$. An important feature of the viewpoint of this paper is that the natural notion of adjacency between objects in an exceptional collection is provided by the underlying quiver of A, rather than the ordering of the collection.

5. Calculating the Derived Tensor Product

To make use of Theorem 2.1 and Proposition 3.1, we need an effective method of calculating the derived tensor product $M \overset{\mathbf{L}}{\otimes}_A N$, for a right A-module M and a left A-module N.

One such method, uses a projective resolution

$$P^d \to \cdots \to P^0 \to A$$

of A in the category of A, A-bimodules, i.e. modules with commuting left and right actions of A. Then $M \overset{\mathbf{L}}{\otimes}_A N$ is represented by $M \otimes_A P^* \otimes_A N$, because $M \otimes_A N = M \otimes_A A \otimes_A N$.

Now, any algebra A, that occurs as the endomorphism algebra of a tilting sheaf T, is a finite-dimensional algebra, naturally described as the quotient of the path algebra of a quiver by an 'admissable' ideal of relations. Such an algebra A has a minimal projective resolution in which

$$P^k = \bigoplus_{i,j=1}^{n} Ae_i \otimes V^k_{i,j} \otimes e_j A$$

where $e_1, \dots, e_n$ are the orthogonal idempotents corresponding to the indecomposable summands $T_1, \dots T_n$ of T, while $V^k_{ij} \cong \operatorname{Tor}^k_A(S_i, S_j)$ is a finite dimensional vector space, with S_i being the simple A-module on which only e_i acts non-trivially. The length d of this resolution is the global dimension of A.

In particular, when T is a tilting bundle, then this means that $T^\vee \overset{\mathbf{L}}{\boxtimes}_A T$ is represented by a complex whose kth term is

$$\bigoplus_{i,j=1}^{n} \pi_1^* T_i^\vee \otimes V^k_{ij} \otimes \pi_2^* T_j.$$

This complex is thus a locally-free resolution of $\mathcal{O}_\Delta$.

For a general description of the minimal resolution of A, see [BK]. Below, we shall just describe the resolution when A has global dimension less than or equal to two, which is (almost) sufficient for the purposes of this paper. First, though, we must explain a little about the description of an algebra and its modules by a quiver and their representations.

Suppose $T_1, \dots, T_n$ are pairwise non-isomorphic indecomposable summands of an object T in an abelian k-category, and $A = \operatorname{End}(T)$. Then it is very natural to consider a right A-module as equivalent to a contravariant k-linear functor from $\{T_i\}$ to the category of k vector spaces. In particular, the A-module $\operatorname{Hom}(T, M)$ is identified with the functor $\operatorname{Hom}(-, M)$ restricted to $\{T_i\}$. Now, we can present the category $\{T_i\}^{op}$ by a quiver with relations. That is to say, we can choose an abstract set Q_0 of vertices corresponding to the objects T_i and a finite set Q_1 of arrows $a : ta \to ha$, where $ta, ha \in Q_0$, representing a maximal set of independent irreducible maps in $\{T_i\}$, but in the opposite sense. Any map can then be written as a linear combination of 'paths' in the quiver, i.e. allowable composites of the arrow maps, subject to a finite set R of relations $\rho : t\rho \cdots h\rho$,

$$\rho = \sum_{i=1}^{k} \lambda_i a_{i1} \dots a_{in_i}$$

where $ta_{i1} = t\rho$, $ha_{ir} = ta_{ir+1}$ for $1 \le r < n_i$, and $ha_{in_i} = h\rho$, for each i. An A-module, or a contravariant linear functor on $\{T_i\}$, is then just a representation of the quiver in the category of k vector spaces for which all the relations evaluate to zero.

We can now describe the minimal resolution of A, as promised.

Proposition 5.1. *Let A be a finite dimensional algebra, described in terms of a quiver Q_0, Q_1 with relations R as above and let $\{e_i \mid i \in Q_0\}$*

be a set of indecomposable orthogonal idempotents. Then the following complex of A, A-bimodules is the final part, i.e. $P^2 \to P^1 \to P^0$, of the minimal projective resolution of A.

$$\bigoplus_{\rho\in R} Ae_{t\rho}\otimes[\rho]\otimes e_{h\rho}A \longrightarrow \bigoplus_{a\in Q_1} Ae_{ta}\otimes[a]\otimes e_{ha}A \longrightarrow \bigoplus_{i\in Q_0} Ae_i\otimes[i]\otimes e_iA$$

where $[\rho]$, $[a]$ and $[i]$ should be interpreted as one-dimensional vector spaces acting as labels. The maps in the sequence are given by

$$\begin{aligned} e_{t\rho}\otimes[\rho]\otimes e_{h\rho} &\mapsto \sum_{i=1}^{k}\lambda_i\sum_{j=1}^{n_i} a_{i1}\dots a_{ij-1}\otimes[a_{ij}]\otimes a_{ij+1}\dots a_{in_i} \\ e_{ta}\otimes[a]\otimes e_{ha} &\mapsto a\otimes[ha]\otimes e_{ha} - e_{ta}\otimes[ta]\otimes a \end{aligned}$$

The map from the last term onto A is $e_i\otimes[i]\otimes e_i\mapsto e_i$.

Proof. See [BK]. □

Clearly, when A has global dimension less than or equal to two, this complex is the whole projective resolution of A.

Example 5.2. To illustrate Proposition 5.1, we consider the tilting bundle

$$T = \mathcal{O}\oplus\mathcal{O}(1)\oplus\mathcal{O}(2)$$

on $\mathbb{P}^2$ (c.f. [Ba]), where $\mathcal{O}(n)$ is the nth power of the hyperplane bundle. Notice that $\mathcal{O},\mathcal{O}(1),\mathcal{O}(2)$ is a strongly exceptional collection. Then $A=\operatorname{End}(T)$ is described by the following quiver with relations

$$\bullet \overset{\overline{x}}{\underset{\overline{z}}{\xrightarrow{\;\overline{y}\;}}} \bullet \overset{x}{\underset{z}{\xrightarrow{\;y\;}}} \bullet \qquad \overline{y}z-\overline{z}y=\overline{z}x-\overline{x}z=\overline{x}y-\overline{y}x=0$$

where $x,y,z:\mathcal{O}\to\mathcal{O}(1)$ and $\overline{x},\overline{y},\overline{z}:\mathcal{O}(1)\to\mathcal{O}(2)$ are the maps given by multiplication by a basis of sections of $\mathcal{O}(1)$. Notice that the arrows in the quiver go in the opposite direction to the maps. In this case $A=\operatorname{End}(T)$ has global dimension 2 and Proposition 5.1 yields the following complex for $T^\vee \overset{\mathbf{L}}{\boxtimes}_A T$

$$3\big(\mathcal{O}(-2)\boxtimes\mathcal{O}\big) \xrightarrow{d_2} \begin{matrix} 3\big(\mathcal{O}(-1)\boxtimes\mathcal{O}\big) \\ \oplus \\ 3\big(\mathcal{O}(-2)\boxtimes\mathcal{O}(1)\big)\end{matrix} \xrightarrow{d_1} \begin{matrix}\mathcal{O}\boxtimes\mathcal{O} \\ \oplus \\ \mathcal{O}(-1)\boxtimes\mathcal{O}(1) \\ \oplus \\ \mathcal{O}(-2)\boxtimes\mathcal{O}(2)\end{matrix}$$

with maps

$$d_1 = \begin{pmatrix} x^{[1]} & y^{[1]} & z^{[1]} & 0 & 0 & 0 \\ -x^{[2]} & -y^{[2]} & -z^{[2]} & \overline{x}^{[1]} & \overline{y}^{[1]} & \overline{z}^{[1]} \\ 0 & 0 & 0 & -\overline{x}^{[2]} & -\overline{y}^{[2]} & -\overline{z}^{[2]} \end{pmatrix}$$

and

$$d_2 = \begin{pmatrix} 0 & \overline{z}^{[1]} & -\overline{y}^{[1]} \\ -\overline{z}^{[1]} & 0 & \overline{y}^{[1]} \\ \overline{y}^{[1]} & -\overline{x}^{[1]} & 0 \\ 0 & -z^{[2]} & y^{[2]} \\ z^{[2]} & 0 & -x^{[2]} \\ -y^{[2]} & x^{[2]} & 0 \end{pmatrix}$$

where $x^{[i]}$ means multiplication by the section $\pi_i^* x$ in the ith factor of the exterior tensor product, i.e. $x^{[1]} = x \boxtimes 1$ while $x^{[2]} = 1 \boxtimes x$.

Using the Euler sequences for the bundles Ω^i of i-forms,

$$0 \to \Omega^1 \to \mathcal{O}(-1)^3 \to \mathcal{O} \to 0$$
$$0 \to \Omega^2 \to \mathcal{O}(-2)^3 \to \Omega^1 \to 0$$

we can reduce this complex to the more familiar form

$$\mathcal{O}(-2) \boxtimes \Omega^2(2) \longrightarrow \mathcal{O}(-1) \boxtimes \Omega^1(1) \longrightarrow \mathcal{O} \boxtimes \mathcal{O}$$

that was given by Beilinson [Be1] as a resolution of $\mathcal{O}_\Delta$. Indeed, it was this resolution that was originally used to show that T is a tilting sheaf.

6. Some Strongly Exceptional Collections

In this section, we describe strongly exceptional collections of line bundles on a number of rational surfaces X, namely the rational ruled surfaces, or Hirzebruch surfaces Σ_n, and the blow-ups $\mathbb{P}^2(m)$ of the projective plane $\mathbb{P}^2$ in at m general points, for $m \leq 3$. Of course, this latter notation is ambiguous for $m \geq 5$, because such blow-ups have moduli, but we will not consider these surfaces here. In the case of Σ_n, these collections have already been described in [KN].

For $n \geq 0$, the Hirzebruch surface $\Sigma_n = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(-n) \oplus \mathcal{O}_{\mathbb{P}^1})$. Such a surface can be embedded in $\mathbb{P}^1 \times \mathbb{P}^{n+1}$ as the subvariety

$$\{(a_0 : a_1), (b_0 : \ldots : b_{n+1}) \mid a_0 b_i = a_1 b_{i-1}, \text{ for } 1 \leq i \leq n\}. \tag{6.1}$$

In particular, $\Sigma_0 = \mathbb{P}^1 \times \mathbb{P}^1$ and $\Sigma_1 = \mathbb{P}^2(1)$. The line bundles on Σ_n are obtained by pulling back the line bundles on $\mathbb{P}^1$ and $\mathbb{P}^{n+1}$ via the embedding and one of the projections π_i. Thus, $\mathrm{Pic}(\Sigma_n) = \mathbb{Z}^2$, with $\mathcal{O}_{\Sigma_n}(p,q) = \pi_1^*\big(\mathcal{O}_{\mathbb{P}^1}(p)\big) \otimes \pi_2^*\big(\mathcal{O}_{\mathbb{P}^{n+1}}(q)\big)$. The canonical bundle is $\mathcal{O}(n-2,-2)$ and the intersection form is given by the matrix $\begin{pmatrix} 0 & 1 \\ 1 & n \end{pmatrix}$.

Proposition 6.1. *Over Σ_n, the line bundles $\mathcal{O}$, $\mathcal{O}(1,0)$, $\mathcal{O}(0,1)$, $\mathcal{O}(1,1)$ are a strongly exceptional collection.*

Proof. We simply need to show that the differences $\mathcal{O}(1,0)$, $\mathcal{O}(0,1)$, $\mathcal{O}(1,1)$ and $\mathcal{O}(-1,1)$ satisfy Condition (ii) of Lemma 4.3. Now, the divisors $(1,0)$ and $(0,1)$ are both moving, so any line bundle with negative degree along either one must have $H^0 = 0$. Hence $H^0(\mathcal{O}(p,q)) = 0$

when $q < 0$ or $p + nq < 0$, and, by Serre duality, $H^2(\mathcal{O}(p,q)) = 0$, when $q > -2$ or $p + nq > -2$. This gives all the required vanishing of H^0 and H^2.

The Riemann-Roch formula gives

$$\begin{aligned} \chi(\mathcal{O}(D)) &= \frac{1}{2} D \cdot (D - K) + 1 \\ &= \frac{1}{2}(2p + nq + 2)(q+1) \end{aligned}$$

Hence, if $D = (-1, 0)$ or $(p, -1)$, for any p, then $H^i(\mathcal{O}(D)) = 0$, for all i. Now, let $C \in |\mathcal{O}(0,1)|$ be a smooth rational curve (e.g. $b_{n+1} = 0$). From the short exact sequence

$$0 \to \mathcal{O}(p, q-1) \to \mathcal{O}(p,q) \to \mathcal{O}_C(p + nq) \to 0,$$

we get a long exact sequence in cohomology

$$\cdots \to H^1\big(\mathcal{O}(p, q-1)\big) \to H^1\big(\mathcal{O}(p,q)\big) \to H^1\big(\mathcal{O}_C(p+nq)\big).$$

Thus, from the vanishing of $H^1\big(\mathcal{O}(-1,0)\big)$, $H^1\big(\mathcal{O}(0,-1)\big)$ and $H^1\big(\mathcal{O}(1,-1)\big)$, we can deduce the rest of the vanishing we need. $\square$

For each such surface $X = \Sigma_n$, the algebra $A = \mathrm{End}_X\big(\mathcal{O} \oplus \mathcal{O}(1,0) \oplus \mathcal{O}(0,1) \oplus \mathcal{O}(1,1)\big)$ is given by one of the following quivers with relations:

Case (i): Σ_0

$$\begin{tikzcd} \bullet \arrow[r, shift left, "\overline{a}_0"] \arrow[r, shift right, "\overline{a}_1"'] \arrow[d, shift right, "\overline{b}_0"'] \arrow[d, shift left, "\overline{b}_1"] & \bullet \arrow[d, shift right, "b_0"'] \arrow[d, shift left, "b_1"] \\ \bullet \arrow[r, shift left, "a_0"] \arrow[r, shift right, "a_1"'] & \bullet \end{tikzcd} \qquad \begin{array}{l} \overline{a}_0 b_0 - \overline{b}_0 a_0 \\ \overline{a}_1 b_1 - \overline{b}_1 a_1 \\ \overline{b}_0 a_1 - \overline{a}_1 b_0 \\ \overline{b}_1 a_0 - \overline{a}_0 b_1 \end{array}$$

Case (ii): Σ_1

$$\begin{tikzcd} \bullet \arrow[r, shift left, "\overline{a}_0"] \arrow[r, shift right, "\overline{a}_1"'] \arrow[dr, "\overline{b}"'] & \bullet \arrow[d, "d"'] \arrow[dr, "b"] & \\ & \bullet \arrow[r, shift left, "a_0"] \arrow[r, shift right, "a_1"'] & \bullet \end{tikzcd} \qquad \begin{array}{l} \overline{a}_1 b - \overline{b} a_1 \\ \overline{b} a_0 - \overline{a}_0 b \\ \overline{a}_0 d a_1 - \overline{a}_1 d a_0 \end{array}$$

Case (iii): Σ_n

$$\begin{tikzcd} \bullet \arrow[r, shift left, "\overline{a}_0"] \arrow[r, shift right, "\overline{a}_1"'] \arrow[dr, "\overline{b}"'] & \bullet \arrow[d, shift right=2, "d_1"'] \arrow[d, shift left=2, "d_n"] \arrow[d, phantom, "\cdots"] \arrow[dr, "b"] & \\ & \bullet \arrow[r, shift left, "a_0"] \arrow[r, shift right, "a_1"'] & \bullet \end{tikzcd} \qquad \begin{array}{l} \overline{a}_1 b - \overline{b} a_1 \\ \overline{b} a_0 - \overline{a}_0 b \\ d_i a_1 - d_{i+1} a_0 \\ \overline{a}_0 d_{i+1} - \overline{a}_1 d_i \end{array}$$

for $n \geq 2$ and for $1 \leq i \leq n-1$. These generating relations overdetermine the relations between the paths joining the extreme vertices by $n-2$ doubly determined relations:

$$\overline{a}_1 d_{j-1} a_1 - \overline{a}_1 d_j a_0 - \overline{a}_0 d_j a_1 + \overline{a}_0 d_{j+1} a_0$$

for $2 \leq j \leq n-1$.

Now, let $X = \mathbb{P}^2(m)$, for $m \leq 3$. The first two cases $\mathbb{P}^2(0) = \mathbb{P}^2$ and $\mathbb{P}^2(1) = \Sigma_1$ have been covered already in Example 5.2 and Proposition 6.1. Let H be the pullback of the hyperplane divisor in $\mathbb{P}^2$ and $E_1, \dots, E_n$ be the exceptional divisors. These form a basis for $\mathrm{Pic}(X)$. The canonical bundle is $\mathcal{O}(-3H + E_1 + \cdots + E_n)$ and the intersection form is diagonal with entries $(1, -1, \dots, -1)$.

Proposition 6.2.
i) On $\mathbb{P}^2(2)$, the line bundles $\mathcal{O}$, $\mathcal{O}(H-E_1)$, $\mathcal{O}(H-E_2)$, $\mathcal{O}(H)$, $\mathcal{O}(2H-(E_1+E_2))$ form a strongly exceptional collection.
ii) On $\mathbb{P}^2(3)$, the line bundles $\mathcal{O}$, $\mathcal{O}(H-E_1)$, $\mathcal{O}(H-E_2)$, $\mathcal{O}(H-E_3)$, $\mathcal{O}(H)$, $\mathcal{O}(2H-(E_1+E_2+E_3))$ form a strongly exceptional collection.

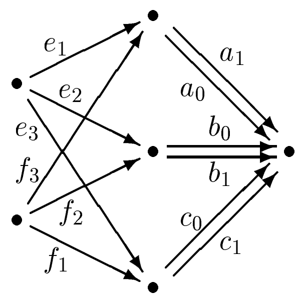
Proof. Working over $\mathbb{P}^2(m)$, for any m, we can verify Condition (ii) of Lemma 4.3 for $\mathcal{O}(E_i)$, for $\mathcal{O}(E_i - E_j)$, when $i \neq j$, and for $\mathcal{O}(A-B)$, when $A = H$ or $2H$ and $B = 0, E_i, E_i + E_j$ or $E_i + E_j + E_k$, with i, j and k distinct. This is done in essentially the same way as Proposition 6.1. Using the moving divisors H and $H - E_i$ and Serre duality, we get all the vanishing of H^0 and H^2. From Riemann-Roch we find in addition that $H^1\big(\mathcal{O}(pH + \sum q_i E_i\big) = 0$, when $p = -1$ or -2 and each $q_i = 0$ or 1. Then we use the structure sequence for smooth rational curves in $|H|$ and $|H - E_i|$ to deduce almost all of the rest of the H^1 vanishing. The one we miss is $\mathcal{O}(H - (E_i + E_j + E_k))$, for which we already know that $H^2 = 0$ and $\chi = 0$. But now, because the images of E_i, E_j and E_k under blowing down are not colinear, we know that $H^0 = 0$ and hence that $H^1 = 0$ also. $\square$

For $X = \mathbb{P}^2(m)$, the algebra $A = \mathrm{End}_X(T)$, where T is the direct sum of the line bundles in the corresponding exceptional collection, is given by one of the following quivers with relations:

Case (iv): $\mathbb{P}^2(2)$

$$\begin{tikzcd}
& & \bullet \arrow[dr, shift left, "a_1"] \arrow[dr, shift right, "a_0"'] & \\
\bullet \arrow[urr, "\overline{b}_0"] \arrow[r, "f"] \arrow[drr, "\overline{a}_0"'] & \bullet \arrow[ur, "e_1"] \arrow[dr, "e_2"'] & & \bullet \\
& & \bullet \arrow[ur, shift left, "b_0"] \arrow[ur, shift right, "b_1"'] &
\end{tikzcd}
\qquad
\begin{array}{l}
f e_2 b_0 - \overline{b}_0 a_0 \\
\overline{a}_1 b_1 - f e_1 a_1 \\
\overline{b}_0 a_1 - \overline{a}_1 b_0 \\
e_1 a_0 - e_2 b_1
\end{array}$$

Case (v): $\mathbb{P}^2(3)$



$$\begin{array}{l} e_2b_0 - e_3c_1 \\ e_3c_0 - e_1a_1 \\ e_1a_0 - e_2b_1 \\ f_2b_1 - f_3c_0 \\ f_3c_1 - f_1a_0 \\ f_1a_1 - f_2b_0 \end{array}$$

7. The Surfaces as Moduli Spaces

In Section 6, we defined algebras A to be the endomorphism algebras of certain vector bundles over the surfaces X. In this section, we show that each such surface X can be recovered from the corresponding algebra A as a moduli space of θ-stable A-modules, in the sense of [Ki]. To specify such a moduli space we must give a dimension vector α, i.e. a non-negative integer α_v for each vertex v of the quiver defining A, and a weight vector or 'character' θ, i.e. an integer θ_v for each vertex, such that $\sum_v \theta_v\alpha_v = 0$. The moduli space of θ-stable A-modules of dimension vector α is then the parameter space for those A-modules which have no proper submodules with any dimension vector β for which $\sum_v \theta_v\beta_v \leq 0$.

In each case, we will choose $\alpha_v = 1$, for all v. The character θ will be depend on the algebra. For book-keeping purposes, the vertices of the quiver inherit an order from the corresponding order of the strongly exceptional collection. We use this order to write θ as a row vector.

For any of the algebras A arising from a Hirzebruch surface (i.e. cases (i), (ii) and (iii) in ection), take $\theta = (1, 0, 0, -1)$. For θ-stable A-modules, the vanishing of certain combinations of arrows is prevented, because such vanishing implies the existence a submodule contradicting the stability condition. More precisely:

vanishing of	$\Rightarrow$	submod. of dim.
$\overline{a}_0, \overline{a}_1$		$(1, 1, 0, 1)$
a_0, a_1		$(0, 1, 0, 0)$

Hence, $(a_0 : a_1)$ and $(\overline{a}_0 : \overline{a}_1)$ are well-defined points in $\mathbb{P}^1$.

Case (i): By symmetry, we also see that $(b_0 : b_1)$ and $(\overline{b}_0 : \overline{b}_1)$ are well-defined points in $\mathbb{P}^1$. The relations then imply that $(a_0 : a_1) = (\overline{a}_0 : \overline{a}_1)$ and $(b_0 : b_1) = (\overline{b}_0 : \overline{b}_1)$, and that, up to equivalence, these two points determine the representation. Hence the moduli space is $\mathbb{P}^1 \times \mathbb{P}^1 = \Sigma_0$.

Case(ii): Set

$$\begin{array}{lll} b_0 = da_0 & b_1 = da_1 & b_2 = b \\ \overline{b}_0 = \overline{a}_0 d & \overline{b}_1 = \overline{a}_1 d & \overline{b}_2 = \overline{b} \end{array}$$

Case (iii): Set

$$\begin{array}{llll} b_0 = d_1 a_0 & b_i = d_{i-1} a_0 = d_i a_1 & b_n = d_n a_1 & b_{n+1} = b \\ \overline{b}_0 = \overline{a}_0 d_1 & \overline{b}_i = \overline{a}_0 d_{i-1} = \overline{a}_1 d_i & \overline{b}_n = \overline{a}_1 d_n & \overline{b}_{n+1} = \overline{b} \end{array}$$

for $i = 1, \dots, n-1$.

In both these cases, $(b_0 : \cdots : b_{n+1})$ and $(\overline{b}_0 : \cdots : \overline{b}_{n+1})$ are well-defined points in $\mathbb{P}^{n+1}$, because there are no submodules of dimension vector $(0,0,1,0)$ and $(1,0,1,1)$. In addition, the relations imply that $(a_0 : a_1) = (\overline{a}_0 : \overline{a}_1)$ and $(b_0 : \cdots : b_{n+1}) = (\overline{b}_0 : \cdots : \overline{b}_{n+1})$, so we have a point in $\mathbb{P}^1 \times \mathbb{P}^{n+1}$. By construction, it satisfies the equations (6.1) and so is a point in Σ_n, as required.

Case (iv): Choose $\theta = (2,0,0,-1,-1)$, and set $x_0 = e_1 a_1$, $x_1 = e_2 b_0$ and $x_2 = e_1 a_0 = e_2 b_1$. Then observe that,

vanishing of	$\Rightarrow$	submod. of dim.
$\overline{b}_0, f$		$(1,0,1,0,1)$
$\overline{b}_0, e_1$		$(1,0,1,1,1)$
$\overline{a}_1, f$		$(1,1,0,0,1)$
$\overline{a}_1, e_2$		$(1,1,0,1,1)$
e_1, e_2		$(0,0,0,1,0)$
a_0, a_1		$(0,1,0,0,0)$
b_0, b_1		$(0,0,1,0,0)$

From this and the relations, $(a_0 : a_1) = (fe_2 : \overline{a}_1)$ and $(b_0 : b_1) = (\overline{b}_0 : fe_1)$ are points in $\mathbb{P}^1$, while $(x_0 : x_1 : x_2) = (\overline{a}_1 e_1 : \overline{b}_0 e_2 : fe_1 e_2)$ is a point in $\mathbb{P}^2$. The point we thereby obtain in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^2$ also satisfies $a_0 x_0 = a_1 x_2$ and $b_0 x_2 = b_1 x_1$, which are the equations of $\mathbb{P}^2$ blown up at $(0:1:0)$ and $(1:0:0)$.

Case (v): Choose $\theta = (2,0,0,0,-1,-1)$, and observe that

vanishing of	$\Rightarrow$	submod. of dim.
a_0, a_1		$(0,1,0,0,0,0)$
e_1, f_1		$(1,0,1,1,1,1)$
e_1, e_2, e_3		$(0,0,0,0,1,0)$

with similar results for (b_0, b_1), (c_0, c_1), (e_2, f_2), (e_3, f_3) and (f_1, f_2, f_3). The relations then imply that two of the e's or f's cannot vanish simultaneously. Therefore $(x_1 : x_3 : x_3) = (f_1 e_2 e_3 : e_1 f_2 e_3 : e_1 e_2 f_3)$ and $(y_1 : y_2 : y_3) = (e_1 f_2 f_3 : f_1 e_2 f_3 : f_1 f_2 e_3)$ give a well-defined point in $\mathbb{P}^2 \times \mathbb{P}^2$ satisfying $x_1 y_1 = x_2 y_2 = x_3 y_3$. This is one description of $\mathbb{P}^2(3)$. The relations imply that the point $(a_0 : a_1), (b_0 : b_1), (c_0 : c_1) \in \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ is determined by $(x_1 : x_3 : x_3), (y_1 : y_2 : y_3)$, so that the moduli space is just $\mathbb{P}^2(3)$.

8. Some Tilting Bundles

In this section, we show that each strongly exceptional collection described in Section 6 actually consists of the indecomposable summands of a tilting bundle. By Proposition 3.1 and Lemma 4.2, we must simply show that each collection can be used to make a locally free resolution of the diagonal sheaf $\mathcal{O}_\Delta$ on $X \times X$, in the manner described in Section 5. We use the following fundamental lemma to help identify such resolutions.

Lemma 8.1. *Let Y be a smooth variety and $0 \to V_d \to \cdots \to V_0$ a complex of vector bundles on Y which is exact over every point outside a subvariety Z of codimension d. Then the complex is exact, as a sequence of sheaves, on the whole of Y, i.e. it is a locally free resolution of the cokernel of $d_1 : V_1 \to V_0$.*

Proof. We prove the result locally. Over any local ring $R = \mathcal{O}_{y,Y}$, we have a complex of free R-modules, which we wish to show is exact. For this we can use the standard necessary and sufficient condition of Buchsbaum and Eisenbud ([BE] or [No] Theorem 6.15). Because the complex is generically exact, the ranks of the differentials are what they must be and because R is regular, and hence Cohen-Macaulay, the determinantal ideal of each differential has depth at least d. This is in fact stronger than what is needed for the complex to be exact. □

This lemma is the essential ingredient in the following more specific result.

Lemma 8.2. *Let X be a smooth variety of dimension d and $V_r \to \cdots \to V_0$ a complex of vector bundles on $X \times X$, with $r \geq d$. Then the complex is a locally free resolution of $\mathcal{O}_\Delta$ if*
i) For $i > d$, it is exact, as a sequence of bundles, over $X \times X$,
ii) For $0 \leq i \leq d$, it is exact, as a sequence of bundles, over $X \times X \smallsetminus \Delta$,
iii) The cokernel of $d_1 : V_1 \to V_0$, at any point $(x, x) \in \Delta$, is canonically isomorphic to the field k of scalars.

Proof. Note that, by Lemma 8.1, the condition $r \geq d$ is necessary. Now, Condition (i) implies that we can shorten the complex of vector bundles to one of length d with the same cohomology. Condition (ii) and Lemma 8.1 show that we have a resolution of some sheaf supported on Δ, while Condition (iii) identifies this sheaf as $\mathcal{O}_\Delta$. □

In the particular case which interests us, this result can be paraphrased in terms of the cohomological properties of the A-modules which are the fibres of our prospective tilting bundle T. Note that, by a "family T of A-modules parametrised by X" we mean a vector bundle T over X and a homomorphism $A \to \mathrm{End}(T)$.

Lemma 8.3. *Let T be a family of A-modules parametrised by a smooth variety X of dimension d. Then $T^\vee \overset{\mathbf{L}}{\boxtimes}_A T$ provides a locally free resolution of $\mathcal{O}_\Delta$, if*
i) For $i > d$, $\mathrm{Ext}^i_A(T_x, T_y) = 0$ for all x, y,
ii) For $1 \leq i \leq d$, $\mathrm{Ext}^i_A(T_x, T_y) = 0$ for all $x \neq y$,

iii) $\mathrm{Hom}_A(T_x, T_y) = \begin{cases} k & \textit{if } x = y \\ 0 & \textit{otherwise} \end{cases}.$

Proof. One simply needs to observe that the fibre over $(y, x) \in X \times X$ of the complex which calculates $T^\vee \overset{\mathbf{L}}{\boxtimes}_A T$ is the complex which calculates the derived functor of $T_y^\vee \otimes_A T_x = \mathrm{Hom}_A(T_x, T_y)^\vee$. □

Now, as observed in Section 7, in each of the cases that interest us X is a moduli space of θ-stable A-modules and T is the universal family over X. Hence, Condition (iii) of Lemma 8.3 is a consequence of Schur's Lemma, since θ-stable modules are simple objects in the full subcategory of θ-semistable modules.

Most of the algebras described in Section 6 have global dimension 2, and hence Condition (i) of Lemma 8.3 is automatically satisfied. However, for $n \geq 3$, the algebra associated to the Hirzebruch surface Σ_n has global dimension 3. In this case, the doubly determined relations

$$D_i = \overline{a}_1 d_{i-1} a_1 - \overline{a}_1 d_i a_0 - \overline{a}_0 d_i a_1 + \overline{a}_0 d_{i+1} a_0$$

contribute a fourth term to the minimal resolution of A (see [BK] for details), namely

$$\bigoplus_{i=2}^{n-1} Ae_1 \otimes [D_i] \otimes e_4 A$$

which is joined onto the complex given in Proposition 5.1 by the map

$$\begin{aligned} e_1 \otimes [D_i] \otimes e_4 \quad \mapsto \quad & \overline{a}_1 \otimes [d_{i-1}a_1 - d_i a_0] \otimes e_4 - \overline{a}_0 \otimes [d_i a_1 - d_{i+1} a_0] \otimes e_4 \\ & + e_1 \otimes [\overline{a}_0 d_i - \overline{a}_1 d_{i-1}] \otimes a_1 - e_1 \otimes [\overline{a}_0 d_{i+1} - \overline{a}_1 d_i] \otimes a_0 \end{aligned}$$

This gives rise to a map $d_3 : V_3 \to V_2$ in the prospective resolution of $\mathcal{O}_\Delta$ where

$$\begin{aligned} V_3 &= (\mathcal{O}(-1,-1) \boxtimes \mathcal{O})^{n-2} \\ V_2 &= (\mathcal{O}(-1,-1) \boxtimes \mathcal{O})^2 \oplus (\mathcal{O}(0,-1) \boxtimes \mathcal{O})^{n-1} \oplus (\mathcal{O}(-1,-1) \boxtimes \mathcal{O}(1,0))^{n-1} \end{aligned}$$

and

$$d_3 = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \overline{a}_1^{[1]} & 0 & \cdots & 0 \\ -\overline{a}_0^{[1]} & \overline{a}_1^{[1]} & \ddots & \vdots \\ 0 & -\overline{a}_0^{[1]} & \ddots & 0 \\ \vdots & \ddots & \ddots & \overline{a}_1^{[1]} \\ 0 & \cdots & 0 & -\overline{a}_0^{[1]} \\ a_1^{[2]} & 0 & \cdots & 0 \\ -a_0^{[2]} & a_1^{[2]} & \ddots & \vdots \\ 0 & -a_0^{[2]} & \ddots & 0 \\ \vdots & \ddots & \ddots & a_1^{[2]} \\ 0 & \cdots & 0 & -a_0^{[2]} \end{pmatrix}$$

This is clearly injective, as a bundle map, since $(a_0, a_1) \neq (0, 0)$. Hence, Condition (i) of Lemma 8.3 is satisfied in this case as well.

Thus, in all cases we have a complex of vector bundles on $X \times X$

$$V_3 \xrightarrow{d_3} V_2 \xrightarrow{d_2} V_1 \xrightarrow{d_1} V_0$$

with d_3 injective (often 0) and $\operatorname{coker} d_1 = \mathcal{O}_\Delta$. Furthermore, one readily checks that

$$\sum_{i=0}^{3} (-1)^i \operatorname{rk} V_i = 0$$

Hence, to show that this complex is a resolution of $\mathcal{O}_\Delta$, we need only to check that, off Δ, the map d_2 has maximal rank. This we now do in each of the five separate cases. Note that, in each case V_2 is a direct sum of line bundles, one for each relation, while V_1 is a direct sum of line bundles, one for each arrow. The line bundles are always exterior products of line bundles on X. Hence, d_2 is a matrix whose entries are maps between the appropriate exterior product line bundles. They will always be induced by multiplication by sections of line bundles on X and we will use the superscripts [1] and [2], as in Example 5.2, to indicate in which factor they act. The columns of the matrices are implicitly indexed by the relations, ordered as in Section 6. The rows of the matrices will be explicitly indexed by the arrows.

Case (i): The map d_2 is

$$\begin{array}{c} b_0 \\ b_1 \\ \overline{b}_0 \\ \overline{b}_1 \\ a_0 \\ a_1 \\ \overline{a}_0 \\ \overline{a}_1 \end{array}\begin{pmatrix} \overline{a}_0^{[1]} & 0 & -\overline{a}_1^{[1]} & 0 \\ 0 & \overline{a}_1^{[1]} & 0 & -\overline{a}_0^{[1]} \\ -a_0^{[2]} & 0 & a_1^{[2]} & 0 \\ 0 & -a_1^{[2]} & 0 & a_0^{[2]} \\ -\overline{b}_0^{[1]} & 0 & 0 & \overline{b}_1^{[1]} \\ 0 & -\overline{b}_1^{[1]} & \overline{b}_0^{[1]} & 0 \\ b_0^{[2]} & 0 & 0 & -b_1^{[2]} \\ 0 & b_1^{[2]} & -b_0^{[2]} & 0 \end{pmatrix}$$

The first 4 rows have rank 4 unless $(\overline{a}_0 : \overline{a}_1)^{[1]} = (a_0 : a_1)^{[2]}$, while the last 4 rows have rank 4 unless $(\overline{b}_0 : \overline{b}_1)^{[1]} = (b_0 : b_1)^{[2]}$. Thus d_2 has rank 4 off Δ.

Case (ii): The map d_2 is

$$\begin{array}{c} a_0 \\ a_1 \\ b \\ \overline{a}_0 \\ \overline{a}_1 \\ \overline{b} \\ d \end{array}\begin{pmatrix} 0 & \overline{b}^{[1]} & -\overline{a}_1 d^{[1]} \\ -\overline{b}^{[1]} & 0 & \overline{a}_0 d^{[1]} \\ \overline{a}_1^{[1]} & -\overline{a}_0^{[1]} & 0 \\ 0 & -b^{[2]} & -da_1^{[2]} \\ b^{[2]} & 0 & -da_0^{[2]} \\ -a_1^{[2]} & a_0^{[2]} & 0 \\ 0 & 0 & \overline{a}_0^{[1]} a_1^{[2]} - \overline{a}_1^{[1]} a_0^{[2]} \end{pmatrix}$$

The first 6 rows have rank 3 unless $(\overline{a}_0 d : \overline{a}_1 d : \overline{b})^{[1]} = (da_0 : da_1 : b)^{[2]}$. On the other hand, the 3rd, 6th and 7th rows have rank 3 unless $(\overline{a}_0 : \overline{a}_1)^{[1]} = (a_0 : a_1)^{[2]}$. Thus d_2 has rank 3 off Δ.

Case (iii): The map d_2 is

$$\begin{array}{c} a_0 \\ a_1 \\ \overline{a}_0 \\ \overline{a}_1 \\ b \\ \overline{b} \\ d_1 \\ d_2 \\ \vdots \\ \vdots \\ d_n \end{array}\begin{pmatrix} 0 & \overline{b}^{[1]} & -d_2^{[1]} & -d_3^{[1]} & \cdots & -d_n^{[1]} & & & & \\ -\overline{b}^{[1]} & 0 & d_1^{[1]} & d_2^{[1]} & \cdots & d_{n-1}^{[1]} & & & & \\ 0 & -b^{[2]} & & & & & d_2^{[2]} & d_3^{[2]} & \cdots & d_n^{[2]} \\ b^{[2]} & 0 & & & & & -d_1^{[2]} & -d_2^{[2]} & \cdots & -d_{n-1}^{[2]} \\ \overline{a}_1^{[1]} & -\overline{a}_0^{[1]} & & & & & & & & \\ -a_1^{[2]} & a_0^{[2]} & & & & & & & & \\ & & -a_0^{[2]} & 0 & \cdots & 0 & -\overline{a}_0^{[1]} & 0 & \cdots & 0 \\ & & a_1^{[2]} & -a_0^{[2]} & \ddots & \vdots & \overline{a}_1^{[1]} & -\overline{a}_0^{[1]} & \ddots & \vdots \\ & & 0 & a_1^{[2]} & \ddots & 0 & 0 & \overline{a}_1^{[1]} & \ddots & 0 \\ & & \vdots & \ddots & \ddots & -a_0^{[2]} & \vdots & \ddots & \ddots & -\overline{a}_0^{[1]} \\ & & 0 & \cdots & 0 & a_1^{[2]} & 0 & \cdots & 0 & \overline{a}_1^{[1]} \end{pmatrix}$$

The last $n+2$ rows have rank $n+2$ unless

$$(\overline{a}_0 : \overline{a}_1)^{[1]} = (a_0 : a_1)^{[2]}.$$

In that case, the last $n+2$ rows have rank n and the first 4 rows, when restricted to the kernel of the last $n+2$ rows, have rank 2 unless

$$(\overline{a}_0 d_1 : \cdots : \overline{a}_0 d_n : \overline{a}_1 d_n : \overline{b})^{[1]} = (d_1 a_0 : \cdots : d_n a_0 : d_n a_1 : b)^{[2]}$$

Thus d_2 has rank $n+2$ off Δ.

Case (iv): The map d_2 is

$$\begin{array}{c} b_0 \\ b_1 \\ \overline{b}_0 \\ e_1 \\ a_0 \\ a_1 \\ e_2 \\ \overline{a}_1 \\ f \end{array} \begin{pmatrix} fe_2^{[1]} & 0 & -\overline{a}_1^{[1]} & 0 \\ 0 & \overline{a}_1^{[1]} & 0 & -e_2^{[1]} \\ -a_0^{[2]} & 0 & a_1^{[2]} & 0 \\ 0 & -f^{[1]}a_1^{[2]} & 0 & a_0^{[2]} \\ -\overline{b}_0^{[1]} & 0 & 0 & e_1^{[1]} \\ 0 & -fe_1^{[1]} & \overline{b}_0^{[1]} & 0 \\ f^{[1]}b_0^{[2]} & 0 & 0 & -b_1^{[2]} \\ 0 & b_1^{[2]} & -b_0^{[2]} & 0 \\ e_2 b_0^{[2]} & -e_1 a_1^{[2]} & 0 & 0 \end{pmatrix}$$

The first 8 rows have rank 4 unless

$$(fe_2 : \overline{a}_1)^{[1]} = (a_0 : a_1)^{[2]} \quad \text{and} \quad (\overline{b}_0 : fe_1)^{[1]} = (b_0 : b_1)^{[2]} \quad \text{(c.f. Case (i))}.$$

On the other hand, omitting the 4th and 7th rows gives a martix with rank 4 unless

$$(\overline{a}_1 e_1 : \overline{b}_0 e_2 : fe_1 e_2)^{[1]} = (e_1 a_1 : e_2 b_0 : e_1 a_0)^{[2]}.$$

Thus d_2 has rank 4 off Δ.

Case (v): The map d_2 is

$$\begin{array}{c} a_0 \\ a_1 \\ b_0 \\ b_1 \\ c_0 \\ c_1 \\ e_1 \\ e_2 \\ e_2 \\ f_1 \\ f_2 \\ f_2 \end{array} \begin{pmatrix} 0 & 0 & e_1^{[1]} & 0 & -f_1^{[1]} & 0 \\ 0 & -e_1^{[1]} & 0 & 0 & 0 & f_1^{[1]} \\ e_2^{[1]} & 0 & 0 & 0 & 0 & -f_2^{[1]} \\ 0 & 0 & -e_2^{[1]} & f_2^{[1]} & 0 & 0 \\ 0 & e_3^{[1]} & 0 & -f_3^{[1]} & 0 & 0 \\ -e_3^{[1]} & 0 & 0 & 0 & f_3^{[1]} & 0 \\ 0 & -a_1^{[2]} & a_0^{[2]} & & & \\ b_0^{[2]} & 0 & -b_1^{[2]} & & & \\ -c_1^{[2]} & c_0^{[2]} & 0 & & & \\ & & & 0 & -a_0^{[2]} & a_1^{[2]} \\ & & & b_1^{[2]} & 0 & -b_0^{[2]} \\ & & & -c_0^{[2]} & c_1^{[2]} & 0 \end{pmatrix}$$

The first 6 rows have rank 5, while the other two 3×3 blocks have rank 2. Thus d_2 has rank 6 unless

$$\begin{aligned}(e_1 f_2 f_3 : f_1 e_2 f_3 : f_1 f_2 e_3)^{[1]} &= (a_1 b_1 : a_0 b_0 : a_1 b_0)^{[2]} \\ (f_1 e_2 e_3 : e_1 f_2 e_3 : e_1 e_2 f_3)^{[1]} &= (a_0 b_0 : a_1 b_1 : a_0 b_1)^{[2]}\end{aligned}$$

But this only happens on Δ.

9. Concluding Remarks

1) Following J. Rickard [Ri], one should extend the notion of a tilting sheaf, to that of a tilting complex in order to allow more general equivalences between $\mathrm{D^b}(\mathcal{O}_X\text{-mod})$ and $\mathrm{D^b}(\text{mod-}A)$.

Definition 9.1. An object $\Omega \in \mathrm{D^b}(A, \mathcal{O}_X\text{-mod})$ is a *two-sided tilting complex* with *inverse* $\widetilde{\Omega} \in \mathrm{D^b}(\mathcal{O}_X, A\text{-mod})$ if and only if

i) $\mathbf{R}\Gamma(\Omega \overset{\mathbf{L}}{\otimes}_X \widetilde{\Omega}) \cong A$ in $\mathrm{D^b}(A, A\text{-mod})$

ii) $\widetilde{\Omega} \overset{\mathbf{L}}{\otimes}_A \Omega \cong \mathcal{O}_\Delta$ in $\mathrm{D^b}(\mathcal{O}_{X\times X}\text{-mod})$.

Since X is smooth, we may assume that Ω and $\widetilde{\Omega}$ are complexes of locally-free sheaves, so that we may replace $\overset{\mathbf{L}}{\otimes}_X$ by $\otimes_X$. We may then also assume, without loss of generality, that $\widetilde{\Omega} = \Omega^\vee$. The following proposition is immediate.

Proposition 9.2. *If Ω is a two-sided locally-free tilting complex with locally-free inverse $\widetilde{\Omega}$, then*

$$- \overset{\mathbf{L}}{\otimes}_A \Omega : \mathrm{D^b}(\text{mod-}A) \to \mathrm{D^b}(\mathcal{O}_X\text{-mod})$$

and

$$\mathbf{R}\Gamma(- \otimes_X \widetilde{\Omega}) : \mathrm{D^b}(\mathcal{O}_X\text{-mod}) \to \mathrm{D^b}(\text{mod-}A)$$

are mutually inverse equivalences of triangulated categories.

2) It is natural to ask what is the property of the surfaces considered in this paper, that enables them to have a tilting bundle. One feature they all share is that they are all toric varieties. This feature is also shared by $\mathbb{P}^n$, but not by some of the other examples such as the flag varieties, on which Kapranov has constructed tilting bundles. However, for the latter examples the summands of the tilting bundle are not all line bundles, whereas for the toric varieties they are. A. Schofield has also described a tilting bundle on $\mathbb{P}^2$ blown up at 4 general points. This surface is not a toric, and indeed the tilting bundle has a summand which is not a line bundle.

These considerations lead us to the following:

Conjecture 9.3. Let X be a smooth complete toric variety. Then X has a tilting bundle whose summands are line bundles.

Pursuing the idea, one might optimistically go as far as:

Conjecture 9.4. Let X be a smooth complete variety that can be written as the geometric quotient of a (Zariski) open subset of a vector space by the linear action of a reductive group. Then X has a tilting bundle.

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